Why Robots should use Fuzzy Mathematical Morphology*

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Abstract: Mobile robots must represent and reason about spatial knowledge acquired from sensor data which are inherently approximate and uncertain. While techniques based on fuzzy sets are increasingly used in this domain, the use of these techniques often rests on intuitive grounds. In this paper, we show that fuzzy mathematical morphology, a theory often used in image processing but mostly ignored in the robotic tradition, can provide a well grounded approach to the treatment of imprecise spatial knowledge in robotics.

Keywords: Fuzzy occupancy grid, fuzzy spatial information, fuzzy mathematical morphology, robotics

1 Introduction

Mobile robots need the ability to represent and reason about spatial knowledge acquired from sensor data, which are inherently approximate and uncertain. In order to make effective use of this knowledge, the robot must deal with this uncertainty in some way. Techniques based on fuzzy sets and fuzzy logic are increasingly used in robotics to address this problem [19, 9]. However, the use of these techniques often rests on intuitive grounds, and their justification is often limited to recording their empirical success.

Image interpretation and processing is another field where the ability to represent and reason about spatial knowledge under imprecision is of paramount importance. In this field, the use of mathematical morphology, and of its fuzzy versions, is recognized as an effective and well founded tool [20, 3, 2]. Interestingly, the arguments for using fuzzy mathematical morphology in image interpretation and processing under imprecision are similar to the ones advocated for the use of fuzzy sets to deal with spatial information in robotics [14]. Moreover, the "occupancy grids" often used as a device to represent spatial knowledge in robotics can be naturally interpreted from an image processing point of view, and they can therefore be subject to the same type of treatment. Curiously enough, this correspondence has seldom been noted.

In this paper, we show that fuzzy mathematical morphology can provide a well grounded approach to represent and reason about imprecise spatial knowledge in robotics. We claim that the two main reasons for using fuzzy mathematical morphology in this context are: (i) the availability of a large set of tools for approximate spatial reasoning, and (ii) the formal grounding of these tools, which provides a clear semantics. We illustrate this claim on three concrete examples: the propagation of positional uncertainty as the robot moves, the extraction of structural information about the environment from imprecise occupancy information, and the assessment of spatial relations. In all cases, we point out the formal properties of fuzzy mathematical morphology that justify its use to address the specific problem.

2 Fuzzy mathematical morphology

We briefly recall the main elements of fuzzy mathematical morphology — see [4] for a more complete account.

Mathematical morphology is originally based on set theory. It has been introduced in 1964 by Matheron [15], in order to study porous media. But this theory evolved rapidly to a general theory of shape and

*This work was partially supported by the Swedish KK Foundation.

Proc. of the 1st Int. ICSC-NAISO Congress on Neuro-Fuzzy Technologies
La Havana, Cuba, January 2002 (to appear)

its transformations, and was applied in particular in image processing and pattern recognition [20]. Additionally to its set theoretical foundations, it also relies on topology on sets, on random sets, on topological algebra, on integral geometry, on lattice theory. Interesting features of mathematical morphology are that it can deal with spatial information at different levels, it can manipulate objects as well as spatial relationships between objects, and it can be used in a numerical setting, in a semi-quantitative one (in particular, in the framework of fuzzy sets), in a purely algebraic setting, and in a symbolic or logical setting.

The four basic operations of mathematical morphology are dilation, erosion, opening and closing. The *dilation* of a set X of $\mathbb{R}^n$ by a set B called *structuring element* is defined as [20]:

$$D_B(X) = \{x \in \mathbb{R}^n \mid B_x \cap X \neq \emptyset\}, \quad (1)$$

where B_x denotes the translation of B at x. Similarly *erosion* of X by B is defined as:

$$E_B(X) = \{x \in \mathbb{R}^n \mid B_x \subset X\}. \quad (2)$$

In these equations, B defines a neighborhood that is considered at each point. It controls the spatial extension of the operations: in particular, the result at a point only depends on a neighborhood of this point defined by the structuring element. It can also be seen as a relationship between points. Then the spatial domain can be considered as a graph where the points are the vertices of the graph, and B defines the edges of the graph (i.e. $y \in B_x$ if x and y are linked by an edge). Edges are not directed if the structuring element is symmetrical (i.e. $y \in B_x \Leftrightarrow x \in B_y$). This interpretation allows to extend the notions of dilation and erosion to any graph. In spatial information processing, the graph can be typically built from regions and adjacency relationships. Morphological operations applied on such graphs therefore allow to deal with any spatial entities, not necessarily points.

¿From these two operators, *opening* and *closing* are defined respectively as $O(X) = D_{\check{B}}[E_B(X)]$ and $C(X) = E_{\check{B}}[D_B(X)]$, where $\check{B}$ denotes the symmetrical of B with respect to the origin of the space. All these operators satisfy a number of algebraic properties, that can be found in [20]. Among the most important ones are commutativity of dilation (respectively erosion) with union or sup (respectively intersection or inf), increasingness[1] of all operators, iterativity properties of dilation and erosion, idempotency of opening and closing, extensivity[2] of dilation (if the origin belongs to the structuring element) and of closing, anti-extensivity of erosion (if the origin belongs to the structuring element) and of opening. We will illustrate in the two next sections how these properties capture the notions of spatial imprecision in robotics.

Several extensions of mathematical morphology have been proposed. In particular, in the following we will make use of fuzzy morphology. Then operations are defined on fuzzy sets (representing spatial entities along with their imprecision) with respect to fuzzy structuring elements. Several definitions of fuzzy mathematical morphology have been proposed (e.g. [4, 8, 21]). Here we just give an example, chosen for its nice properties with respect to classical morphology, where dilation and erosion of a fuzzy set μ by a structuring element ν are respectively defined, for all $x \in \mathbb{R}^n$, by:

$$D_\nu(\mu)(x) = \sup\{t[\nu(y-x), \mu(y)],\ y \in \mathbb{R}^n\}, \quad (3)$$
$$E_\nu(\mu)(x) = \inf\{T[c(\nu(y-x)), \mu(y)],\ y \in \mathbb{R}^n\}. \quad (4)$$

where t is a t-norm and T the associated t-conorm with respect to the complementation c. In these equations, fuzzy sets are assimilated to their membership functions. Properties are similar as in the crisp case and are detailed in [4, 16].

Through the notion of structuring element, mathematical morphology is a powerful tool for dealing with local or regional spatial context. But it has also some features that allow to include more global information, which is particularly important when the spatial arrangement of objects in a scene has to be assessed.

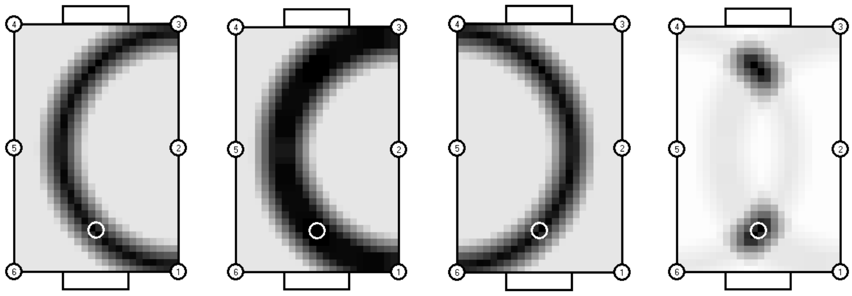
3 Spatial location

We now show some examples of application of fuzzy mathematical morphology to encode spatial uncertainty in the domain of mobile robotics. The first example deals with the uncertainty in the spatial location of objects in the space.

Several popular spatial representations used in mobile robots are based on a discretization of space in a regular 2D grid, where the geometrical position and extent of the relevant objects in the environment are represented. Typical examples include *occupancy grids* [11, 17], representations of the empty and occupied portions of space in the environment; and *locational grids* [6, 7], representations of the robot's location in the environment. Both types of grids accommodate the uncertainty and imprecision in spatial knowledge by associating to each cell a degree, usually interpreted as a probability (probabilistic grids) or as a membership degree (fuzzy grids). In the case of

[1] An operation ψ is increasing if $\forall X, Y\ X \subset Y \Rightarrow \psi(X) \subset \psi(Y)$.

[2] An operation ψ is extensive if $\forall X,\ X \subset \psi(X)$ and anti-extensive if $\forall X, \psi(X) \subset X$.



(a) (b) (c) (d)

Figure 1: Self-localization on a fuzzy position grid. Darker cells have higher possibility. The white circle marks the real position of the robot. (a) The robot observes landmark number 2. (b) Fuzzy dilation after the robot moves in an unknown direction. (c) The robot observes landmark number 5. (d) The observation is fused with the existing information.

fuzzy occupancy grids, for instance, this degree measures, on a $[0, 1]$ scale, the possibility that the area of space corresponding to a given cell is occupied by some object. In the case of fuzzy locational grid, this value measures the possibility that the robot is actually located at that cell.

An interesting, and often un-noticed feature of grid-based representations is that we can easily apply to them techniques typically used for image processing. In particular, we can apply (fuzzy) mathematical morphology, and profit from its ability to explicitly introduce and manipulate imprecision in the representation. Fuzzy dilation has several useful features in this context, including the notion of structuring element, and the properties of extensivity, increasingness, iterativity and commutativity with union (or supremum).

Imprecision can be represented by fuzzy structuring elements. This idea has been developed in [5], where parameters defining positions and orientations are represented as fuzzy numbers, from which the fuzzy structuring elements are derived. For instance, let us assume that a sensor measure provides a point location in the map. From imprecision attached to this measure, it happens that other points can have a non zero possibility degree. Such points are spread around the initial point, forming a fuzzy region. This region can be considered as a fuzzy structuring element centered at this point, and equivalently as the dilation of this point by the fuzzy structuring element (this follows directly from the definition of dilation). The shape of the structuring element can be derived from the sensor model.

The extensivity property captures the expected behavior of imprecision: when imprecision is added, the fuzzy set representing the possible locations becomes larger. Increasingness of dilation with respect to the structuring element is also an important property here: a larger structuring element means more spatial imprecision, and results in a larger dilated set.

An interesting case of imprecise spatial location is given by the robot self-localization problem. Here, the object to localize is the robot itself. In the approach proposed by Buschka *et al* in [7], the robot's belief in its position in the environment is represented by a possibility distribution over the set of possible positions, represented by a 2D grid — see Fig. 1. (This is a simplified view of the 3D representation used in [7].) The information in the grid can represent, and track, multiple possible positions where the robot might be. Maintenance of position information follows the typical *predict-observe-update* cycle of recursive state estimators. The observation of a landmark at a known position is converted to a possibility distribution on the grid that represents the areas where the robot can possibly be given the observation. This distribution is used to update the present position estimate by fuzzy intersection.

The predict part of the cycle is where fuzzy dilation is used. The shape of the structuring element encodes the robot's knowledge of the amount and direction of its own motion. If the estimate of this motion does not include a specific direction, the structuring element will be centered at the origin of the space, and the modal value will be at the origin. If an estimate of the direction of motion is available, we use a shifted structuring element (the modal value is then no more at origin). Performing a dilation with such a structuring element will then combine imprecision and directional shift. In the example treated in [7], the robot is a legged robot: in this case, knowledge about the robot's motion is extremely imprecise since the robot's displacement is subject to the unpredictable slippage of the legs on the ground. Correspondingly, the structuring element is almost circular, with a small bias in the expected direction of motion.

We should note two important advantages of using fuzzy dilation to introduce spatial imprecision. First, the approach is applicable in any dimension and on both crisp and fuzzy objects. For instance if the spatial extension of the object itself is imprecise, this imprecision can be combined with the imprecision on the location or orientation of this object also using a fuzzy dilation. The dilation is then performed on the fuzzy object (where fuzziness represents the imprecision in the spatial extension or delineation of the object) with a fuzzy structuring element representing imprecision on location. The second advantage is that position update by fuzzy dilation is extremely efficient since it relies on local computations on a limited neighborhood of each cell in the position grid.

Two properties of dilation are very important for practical use of the proposed approach. Dilation commutes with fuzzy set union (expressed here by a maximum of the membership degrees), which is written, for two fuzzy sets μ and μ' and a fuzzy structuring element ν, as:

$$D_\nu[\max(\mu, \mu')] = \max[D_\nu(\mu), D_\nu(\mu')]. \tag{5}$$

This allows the direct computation of the dilation of a set of points by the same structuring element. This property is particularly useful if the imprecision induced by some measure is invariant under translation. Then the same structuring element can be used for representing this imprecision all over the space, and instead of dilating each point separately, only one dilation of the whole set of points (or object, or fuzzy object) can be performed. Similarly if the environment contains several objects that are subject to the same imprecision, introducing this imprecision by means of a dilation can be performed in one step on the whole map. Objects do not need to be considered separately.

The second important property is the iteration property of fuzzy dilation, expressed, for a fuzzy set μ and two structuring elements ν and ν', as:

$$D_{\nu'}[D_\nu(\mu)] = D_{D_{\check{\nu}'}(\nu)}(\mu). \tag{6}$$

This property allows one to perform dilation of a fuzzy set successively by two fuzzy structuring elements or equivalently by their dilation. This property is particularly useful when different types of imprecision are represented by different structuring elements. It proves the equivalence between modeling each source of imprecision separately as different structuring elements, and globally as only one structuring element.

The interest of applying successive dilations relies in the possible separation on positional (translation) imprecision, that is likely to be the same everywhere, and on orientation (rotation) imprecision, which may depend on the spatial location, on distance to landmarks, etc. Then both properties expressed by Equations 5 and 6 are directly used. This property is clearly essential to account for successive movements of the robot in our previous self-localization example.

4 Spatial objects

As a second application of fuzzy mathematical morphology to spatial representations in robots, we show its use to extract structural information about the environment from occupancy grids. Occupancy grids are a useful representation for planning and executing collision-free motions, but they do not have any structure that allow the robot to perform higher level reasoning about the environment, e.g., in terms of rooms, doors, and corridors. Mathematical morphology can be used to analyze and extract morphological information from fuzzy occupancy grids built by the robot from uncertain sensor data.

As an example, Fabrizi and Saffiotti [12] use fuzzy mathematical morphology to extract the high level topology of an indoor environment in terms of a graph of rooms and corridors. The authors start from an occupancy grid built from sonar data using the technique described in [17]. Fig. 2 (a) shows a grid built by a Nomad 200 robot in an environment of 18×9 meters consisting of two parallel corridors merging into a large hall, a small area connected to the hall, and two cluttered rooms. Each cell represents a square of side 10 cm. This grid encodes the free space in the environment: each cell stores a number representing, in the $[0, 1]$ interval, the degree of possibility of that part of space being empty. White cells have received evidence of being empty; darker cells have not — they are either occupied or unexplored. (A dual grid, not used here, represents the occupied space.)

The main *spatial objects* of interest here are open spaces (rooms and corridors) connected by narrow passages (doors). The open spaces can be extracted from the grid by performing a morphological *opening* (an erosion followed by dilation) by a fuzzy structuring element of a conic shape that represents the fuzzy concept of a large space. The result, shown in Fig. 2 (b), is a new fuzzy occupancy grid where the value of each cell represents how much each cell belongs to a large open space. Notions from fuzzy digital topology [18] can then be used to partition the transformed grid into connected components, that is, into spatial objects — Fig. 2 (c). The adjacency rela-

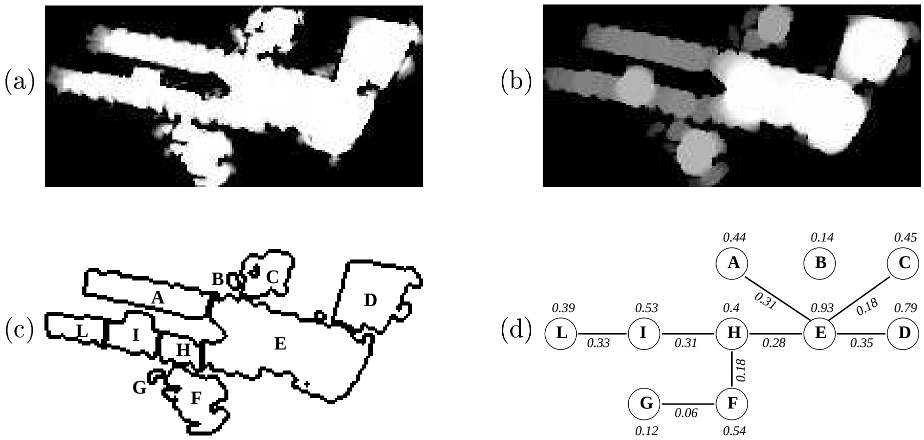


Figure 2: Extracting the topological structure of the environment from an occupancy grid by mathematical morphology. (a) Initial grid. (b) Morphological opening. (c) Connected components. (d) Adjacency graph.

tion between objects can finally be summarized in a graph, as shown in Fig. 2 (d). This graph provides an abstract representation that captures the structure of the space with a reduced number of parameter.

This method as well as the obtained results are fully justified by the formal properties of opening. Opening can be written as:

$$O_B(X) = \{x \mid \exists y \text{ s.t. } x \in B_y \text{ and } B_y \subset X\}, \quad (7)$$

which expresses the fact that opening results in all parts of X that can be covered by translations of B without intersecting the complement of X. Therefore, if the structuring element cannot enter narrow parts (i.e. smaller than the structuring element), then these parts are suppressed (anti-extensivity property of opening) and may induce disconnections. For instance if two large rooms are separated by a narrow pathway, then the rooms become two connected components. The high level structure can therefore be obtained by a topological analysis, in terms of discrete connectivity. In the fuzzy case, similar interpretations can be given and all the above holds to some degree, therefore calling for fuzzy topology.

Idempotency of opening guarantees that once this analysis has been performed using a given structuring element, applying the opening again with the same structuring element does not change the result. This property, along with increasingness of opening with respect to X, captures the idea of filtering small parts. Decreasingness with respect to the structuring element guarantees that the parts that are suppressed are larger if a larger structuring element is used. Therefore the high level structure can be constructed at different scales, where the scale is defined by the size of the structuring element, and captures the notion of "small". Using a conic fuzzy structuring element uses this property, and captures the fact that larger areas have lower degrees of being small.

5 Spatial relations

Spatial relationships between the objects in the environment carry structural information about the environment, and provide important information for object recognition and for self localization [14]. Fuzzy mathematical morphology can be used here to represent and compute in a uniform setting several types of relative position information, like distance, adjacency and directional relative position. In this section, we give a hint of how we can use it to deal with directional relations.

We denote by $\mathcal{S}$ the Euclidean space where the objects are defined. In the case of robot gridmaps, $\mathcal{S}$ is a 2D discrete space. We then consider a (possibly fuzzy) object R, and denote by $\mu_\alpha(R)$ the fuzzy subset of $\mathcal{S}$ such that points of areas which satisfy to a high degree the relation "to be in the direction $\vec{u}_\alpha$ with respect to reference object R" have high membership values. The form of $\mu_\alpha(R)$ may depend on the application domain. Here, we use the definition proposed in [1], which considers those parts of the space that are visible from a reference object point in the direction $\vec{u}_\alpha$. It can be expressed formally as the fuzzy dilation of μ_R by ν, where ν is a fuzzy structuring element depending on α: $\mu_\alpha(R) = D_\nu(\mu_R)$ where μ_R is the membership function of the reference object R. This definition applies both on crisp and fuzzy objects and behaves well even in case of objects with strong concavities [1]. In polar coordinates, ν is defined as: $\nu(\rho, \theta) = f(\theta - \alpha)$ and $\nu(0, \theta) = 1$, where $\theta - \alpha$ is defined modulo π and f is a decreasing function, e.g. $f(x) = \max[0, 1 - \frac{2}{\pi}x]$ for $x \in [0, \pi]$, as illustrated in Figure 3. Algorithms for reducing the computation cost have been proposed in [1].

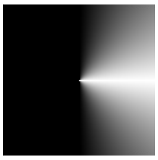


Figure 3: Structuring element ν for $\alpha = 0$ (high grey values correspond to high membership values).

Figure 4 illustrates this definition on the gridmap used in Section 4. The figure shows the fuzzy landscapes for the fuzzy notions of being West, North, East, and South of region E from Fig. 2 (c).

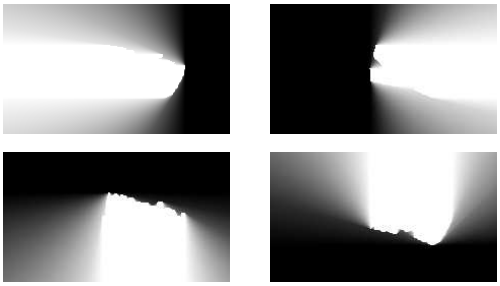


Figure 4: Fuzzy landscapes for being West, East, South and North of region E.

Once we have defined $\mu_\alpha(R)$, we can use it to define the degree to which a given object A is in direction $\vec{u}_\alpha$ with respect to R. Let us denote by μ_A the membership function of the object A. The evaluation of relative position of A with respect to R is given by a function of $\mu_\alpha(R)(x)$ and $\mu_A(x)$ for all x in $\mathcal{S}$. An appropriate tool for defining this function is the fuzzy pattern matching approach [10]. Following this approach, the evaluation of the matching between two possibility distributions consists of two numbers, a necessity degree N (a pessimistic evaluation) and a possibility degree Π (an optimistic evaluation), as often used in the fuzzy set community. In our application, they take the following forms:

$$\Pi_\alpha^R(A) = \sup_{x \in \mathcal{S}} t[\mu_\alpha(R)(x), \mu_A(x)], \quad (8)$$

$$N_\alpha^R(A) = \inf_{x \in \mathcal{S}} T[\mu_\alpha(R)(x), 1 - \mu_A(x)], \quad (9)$$

where t is a t-norm and T a t-conorm. The possibility corresponds to a degree of intersection between the fuzzy sets A and $\mu_\alpha(R)$, while the necessity corresponds to a degree of inclusion of A in $\mu_\alpha(R)$. These operations can also be interpreted in terms of fuzzy mathematical morphology, since $\Pi_\alpha^R(A)$ is equal to the dilation of μ_A by $\mu_\alpha(R)$ at the origin of $\mathcal{S}$, while $N_\alpha^R(A)$ is equal to the erosion at the origin [4]. The set-theoretic and the morphological interpretations indicate how the shape of the objects is taken into account.

The defined directional relations are symmetrical (only for Π), invariant with respect to translation, rotation and scaling, both for crisp and for fuzzy objects, and when the distance between the objects increases, the shape of the objects plays a smaller and smaller role in the assessment of their relative position [1].[3]

It should be emphasized that, since the aim of these definitions is not to find only the dominant relationship, an object may satisfy several different relationships, for different angles, with high degrees. Therefore, "to be to the right of R" does not mean that the object should be completely to the right of the reference object, but only that it is at least to the right of some part of it.

As an illustration, the following table shows, for each region in Fig. 2 (c), the degrees of possibility and necessity of being, respectively, North, South, West and East of region E. Degrees are written as $[N, \Pi]$.

	West	East	South	West
A	[0.88, 1.0]	[0.0, 0.18]	[0.0, 0.80]	[0.35, 1.0]
B	[0.74, 0.87]	[0.39,0.97]	[0.0, 0.0]	[1.0, 1.0]
C	[0.51, 0.87]	[0.48, 1.0]	[0.0, 0.08]	[1.0, 1.0]
D	[0.0, 0.87]	[0.85, 1.0]	[0.0, 0.58]	[0.58,0.89]
E	[1.0, 1.0]	[1.0, 1.0]	[1.0, 1.0]	[1.0, 1.0]
F	[0.88, 1.0]	[0.0, 0.62]	[0.67, 1.0]	[0.0, 0.15]
G	[1.0, 1.0]	[0.0, 0.0]	[0.56, 0.65]	[0.07, 0.12]
H	[1.0, 1.0]	[0.0, 0.0]	[0.41, 0.98]	[0.15, 0.91]
I	[1.0, 1.0]	[0.0, 0.0]	[0.13, 0.52]	[0.15, 0.50]
L	[1.0, 1.0]	[0.0, 0.0]	[0.08, 0.28]	[0.15, 0.29]

These results correspond well to the intuition. For instance, all regions are found to the west of E to some degree, except D which is more ambiguous (as shown by the width of the interval $[N, \Pi]$). A part of D is to the west of E, but all parts of D are to the East of some parts of E. The three regions H, I, L are completely to the West of E, and not at all to the East. They cannot be differentiated using these relations. South and North lead to some differences, and clearly some distance information would completely disambiguate these regions. This information can be used, for instance, to perform topological self-localization during navigation.

Finally, the directed graph in Figure 5 shows the dominant relation for each pair of adjacent regions.

[3] These definitions and properties extend to the 3D case.

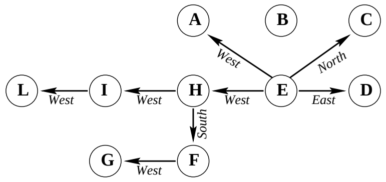


Figure 5: Dominant directional relations from Fig. 2.

6 Conclusion

Fuzzy image processing methods, and in particular fuzzy mathematical morphology, can be profitably used for dealing with imprecise spatial information in autonomous robotics. Importantly, the adequacy of these methods to deal with the specific problems can be justified in terms of the formal properties of the morphological operators. We have shown three examples in which fuzzy mathematical morphology exhibits interesting properties for both knowledge representation and reasoning in robotics: the propagation of positional uncertainty as the robot moves; the extraction of structural information about the environment from imprecise occupancy information; and the addition of relative directional information to it.

There are other aspects of spatial reasoning that lend themselves to a formal treatment using fuzzy mathematical morphology. For instance, robots may employ spatial symbolic reasoning in order to use higher-level knowledge for self-localization, strategic planning, situation assessment, autonomous navigation and multi-robot coordination [13, 14]. In this context, the ability of fuzzy mathematical morphology to ground high-level concepts like "room", "door", and "connected" into sensor data provides a key to combine symbolic, abstract reasoning with quantitative, sensor-based information. This aspect will be investigated in our future work.

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